An Effective Partial Fraction Decomposition Method for Undergraduate Students

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ABSTRACT

In order to propose a more effective alternative method employed by undergraduate students to compute the partial fraction decomposition (PFD) of rational functions which could eventually help to solve the prominent common errors made by undergraduate students during the computation using standard method, data from quizzes and questionnaires administered to 47 students were quantitatively analysed. The findings of this study reveal that there is progress shown in students' performance when they apply the proposed alternative method, namely, the improved cover-up method. Students also appreciate the efficiency and accuracy of the alternative method offered. Hence, it is believed that this approach, which is considerably new to undergraduate students particularly in a public university in Sarawak, can serve as an alternative to the more conventional or standard method of undetermined coefficients used in teaching elementary integral calculus for better students' performance. Additionally, this preliminary evidence provides directions for future research related to partial fraction decomposition methods.

***Keywords**: Partial fraction decomposition (PFD); Improved cover-up method; Undergraduate students*

INTRODUCTION

Partial fraction decomposition (PFD) is the main tool to evaluate integrals of rational functions and rational expressions introduced in elementary integral calculus. PFD is the process of decomposing a complex rational fraction into the sum of simple rational fractions. In other words, PFD is the reverse of the summation of simple rational fractions. Generally, integrating simple rational fractions is usually easier than integrating complex rational functions. Although numerous algorithms or approaches to decompose certain types of rational functions into partial fractions are available, generally, students are introduced to the standard PFD method to solve PFD coefficients. However, it is found that during the computation process using this standard method, students often make common errors leading to incorrect answers or having marks deducted due to the mistakes made. Hence, in this paper, an effective method of PFD is proposed to minimize the tendency of students making arithmetic errors as this will eventually improve their academic performance. This is in line with Stewart and Reeder's (2017) view that students' mathematics performance could be maximized to enhance their academic success and opportunities. Additionally, using this method has the potential to increase the speed while ensuring the accuracy of computations by substitution and division of integer computations for rational number computations (Man, 2009). Choosing and introducing this alternative instructional decision to teach PFD also align with Kraft and Grace's (2016) teaching belief that advocates the importance of skilful selection of teaching strategies to communicate mathematics.

LITERATURE REVIEW

Several computation methods of decomposing a rational function into partial fraction have been broadly employed in the application of calculus, differential equations, control theory and some areas of pure or applied mathematics (Kwang & Xin, 2018; Manoj, Ashvini & Hole, 2020; Kim & Lee, 2016; Ma, Yu & Wang, 2014; Bradley & Cook, 2012; Man, 2011). However, it is observed that two partial fraction decomposition (PFD) coefficients computation methods are more commonly used, namely, the undetermined coefficients method and the cover-up method. According to

Linner (1974, cited in Ma et al., 2014), the well-known cover-up method always serves as a basis for other PFD methods and provides a compact solution to PFD problems. This, however, has a limitation when it comes to the evaluation of high-order poles in high-order polynomials as it could lead to large numerical errors when increasing the successive differentials procedure (Ma et al., 2014; Man, 2009). Another standard PFD method, namely, the undetermined coefficients method, requires the construction of solving the least common denominators by factorization and solution of a system of linear equations during the process of computation. It can be a very lengthy computation, complicated and inconvenient when decomposing more than two partial fractions (Wang, 2011; Gupta, 2011). Therefore, there is a high possibility for students to make more arithmetic mistakes.

The reviews of theoretical and empirical literature show that many students have difficulties solving questions related to the concept of fraction and algebraic expression. In Titus' (2010) research, he found that 35% to 42% of the college students enrolled in development mathematics course committed error patterns in the real number computations. His findings indicate that most of the students have unclear understanding of signed number arithmetic, fractions, distributive property, and exponential errors. Additionally, Brown and Quinn (2006) also discovered that more than fifty percent of 143 ninth graders enrolled in an elementary algebra course at an upper middle-class school, had a lack of experience and proficiency in both fraction concepts and computations. Not only that, this difficulty in solving questions related to the concept of fraction and algebraic expression seems to persist into post-secondary education. According to Hanson and Hogan (2000) who studied the computational estimation skills of 77 college students majoring in a variety of disciplines found that many of the students struggled and were frustrated with the process of finding common denominators. They also noted that a few students in the lower performing groups added or subtracted the numerators and denominators failed to find common denominators. Moreover, Steen (2007) reported that even many adults were confused if a problem requires anything in the simplest of fractions. Considering the above findings, students' difficulties with fractions have led many instructors to opt for alternative methods that could increase the accessibility of PFD method for students with weaknesses related to fraction.

Another error pattern, namely, difficulties with arithmetic, rational number computation and algebra in schools has also been well-documented (Booth, Newton, & Twiss-Garrity, 2014; Ashlock, 2010). In Booth, Newton, and colleagues' (2014) research, the in-depth analysis of school-aged students' errors in algebra problem solving had revealed six common errors made: variable errors, negative sign errors, equality or inequality errors, operation errors, mathematical properties errors, and fraction errors. Moreover, Ashlock (2010) in his analysis of error patterns made by students found that, school-aged students often have misconceptions and make procedural errors in both mathematical operations and methods of computations. These error analyses could help instructors to identify more effective PFD methods in teaching integrals of rational functions and rational expressions.

In view of the above discussions, many PFD methods were proposed to complement the standard methods commonly employed by instructors. Some of the methods are found to perform better than standard methods under specific conditions. For instance, some methods are more suitable for small-scale problems, but they may become complicated when used for the large-scale problems. In Man's (2007) research, he proposed a Heaviside's cover-up method, which requires simple substitutions to find partial fraction coefficient with single poles and apply successive differentiation for multiple poles. Years later, Man (2009) proposed an improved version of Heaviside's approach to compute partial fraction coefficients by using simple substitutions and polynomial divisions. This approach does not involve solving the complex roots of the quadratic polynomial, differentiation, or the solution of a system of linear equations for the PFD of general rational function. Nevertheless, due to its simplicity and useful applications in applied and engineering mathematics as well as the recommendations from other researchers (e.g. Man, 2009 & Manoj et al., 2020) to employ this improved method in teaching integrals of rational functions and rational expressions, this current study aims to explore the potential application of this method on teaching undergraduate students as an alternative method to the undetermined coefficients method in finding PFD coefficients. Towards this end, this study will first analyse and identify the prominent errors made by students when finding the PFD coefficients using the undetermined coefficients method.

PARTIAL FRACTION DECOMPOSITION

A brief explanation of a partial fraction decomposition is presented below:

Let F be a constant field and a(x) and b(x) are linear polynomials in F(x). A proper rational function is $\mathrm{F}(x) = \dfrac{\mathrm{a}(x)}{\mathrm{b}(x)}$, where degree of a(x) is lower than degree of b(x) and $\mathrm{b}(x) = \prod (x - \beta_i)^{m_i}$, β_i are constants, never be 0 and belong to F. m_i, i, k, j and s are positive integers with i, k and j greater than 0 and no more than s.

A partial fraction decomposition of F(x) is:

$$\mathrm{F}(x) = \sum_{i=1}^{s} \left(\frac{\beta_{i,m_1}}{(x - \beta_i)^{m_1}} + \frac{\beta_{i,m_2}}{(x - \beta_i)^{m_2}} + \ldots\ldots \frac{\beta_{i,m_i}}{(x - \beta_i)^{m_i}} \right)$$

, where β_{i,m_i} are coefficient constants

Two methods of PFD were used in this study to compute the unknown coefficients β_{i,m_i} and followed the procedure as shown below:

Method I: The Method of Undetermined Coefficients

Having reconstruction of the combination and the cancellation of least common denominator, applying substitution or equating the coefficients of terms involving the powers of x to solve unknown coefficients β_{i,m_i} , $\beta_{i,m_{i-j}}$ and so on in the system of linear equations until all the unknown coefficients $\beta_{i,m_{i-j}}$ are found.

Method II: The Modified Heavyside's Approach

Firstly, find the first m$_i$-coefficient of $\beta_{i,m_i} = \dfrac{\mathrm{a}(x)}{\mathrm{b}(x)} (x - \beta_i)^{m_i} \big|_{x=\beta_i}$ before the known partial fractions are subtracted from F(x) to find the next m$_{i-j}$-coefficient of

$\beta_{i,m_{i-j}} = \frac{a(x)}{b(x)} - \sum_{k=0}^{j-1} \frac{\beta_{i,m_{i-k}}}{\left(x-\beta_i\right)^{m_{i-k}}} \left(x-\beta_i\right)^{m_{i-j}} |_{x=\beta_i}$. Then, simplify it in polynomial division to become a new function. Apply the same technique to compute the new functions until all the unknown coefficients $\beta_{i,m_{i-j}}$ are found.

The two problems below were used in this study to describe the ways to obtain answers to the questions using Method II.

Problem 1

Find the partial fraction expansion of the rational function, $F(x) = \frac{3x-2}{8x^2+2x-1}$.

Solution:

The PFD express as $\frac{a}{2x+1} + \frac{b}{4x-1}$ where a and b are unknown coefficients to be determined.

The improved cover-up method is used to find a and b as follows:

$$a|_{x=\frac{-1}{2}} = \frac{(3x-2)(2x+1)}{(4x-1)(2x+1)} = \frac{3\left(\frac{-1}{2}\right)-2}{4\left(\frac{-1}{2}\right)-1} = \frac{7}{6}$$

$$b|_{x=\frac{1}{4}} = \frac{(3x-2)(4x-1)}{(2x+1)(4x-1)} = \frac{3\left(\frac{1}{4}\right)-2}{2\left(\frac{1}{4}\right)+1} = \frac{-5}{6}$$

Thus, $F(x) = \frac{-5}{6(2x+1)} + \frac{7}{6(4x-1)}$.

Problem 2

Find the partial fraction expansion of the rational function, $\mathrm{F}(x)=\dfrac{x^2-2x+1}{x(x^2-4)}$.

Solution

The PFD express as $\dfrac{a}{x}+\dfrac{b}{x+2}+\dfrac{c}{x-2}$ where a, b and c are unknown coefficients to be determined.

The improved cover-up method is used to find a, b and c as follows:

$$a\,|_{x=0}=\frac{\left(x^2-2x+1\right)x}{\left(x^2-4\right)x}=\frac{0^0-2\left(0\right)+1}{0^0-4}=\frac{-1}{4}$$

$$b\,|_{x=-2}=\frac{\left(x^2-2x+1\right)\left(x+2\right)}{x\left(x+2\right)\left(x-2\right)}=\frac{\left(-2\right)^2-2\left(-2\right)+1}{-2\left(-2-2\right)}=\frac{9}{8}$$

$$c\,|_{x=2}=\frac{\left(x^2-2x+1\right)\left(x-2\right)}{x\left(x+2\right)\left(x-2\right)}=\frac{\left(2\right)^2-2\left(2\right)+1}{2\left(2+2\right)}=\frac{1}{8}$$

Thus, $\mathrm{F}(x)=\dfrac{-1}{4x}+\dfrac{9}{8\left(x+2\right)}+\dfrac{1}{8\left(x-2\right)}$.

The following section presents the methodology and data collection procedures to meet the objective of this study, specifically, to examine the potential application of the improved cover-up method in teaching undergraduate students as an alternative method to the undetermined coefficients method in learning PFD coefficients.

METHODOLOGY AND PROCEDURE

Participants

The subjects of this study comprised 47 undergraduate students from the Faculty of Civil Engineering in a public university in Sarawak. They were semester two students who enrolled in the Engineering Mathematics class. 25 of them were in class A while the other 22 in class B. Based on the academic background for Calculus 1 and arrangement from the faculty, students who obtained grade C to B+ were grouped in class A, whereas students who obtained grade A- and above were grouped to class B.

Procedure

A total of two quizzes were used to identify error patterns as well as to determine the students' performance. Quizzes were adapted from Item Bank System (IBS) of the institution of higher learning system and were validated by two experienced lecturers who are experts in this subject. Each quiz has two problems related to proper rational functions. The first problem presented PFD with two non-repeated linear factors and the second problem presented PFD with three non-repeated linear factors as shown in the examples provided under the theoretical background presented earlier on. These quizzes testing students' knowledge on the application of Method I and Method II were administered to both the classes, A and B. Students were encouraged to show all their works. They were given 35 minutes to solve each quiz. The students' answer scripts were then collected after they had completed solving the quizzes by using both Method I and Method II.

The scoring rubric for standard PFD method was adapted from IBS and some adjustments were made in accordance with the review feedback obtained from two experienced lecturers who are experts in this subject and literature (Betsy, Kasturi, Chiang, & Goh, 2015). Each step in the solution given was marked and the marks given were allocated for (a) calculation accuracy and mastery of the method concerned (within the range of 0 to 3 marks, where 0 is totally incorrect and 3 is correct without any mistake); (b) number of correct partial fraction coefficients obtained (within the range of 0 to 7 marks, where 0 is incorrect and 7 is correct without any mistake). The allocation of marks in (a) is determined by the total marks in (b). Allocating 0-2 marks for calculation accuracy, 1 mark was awarded for mastery of the method which requires providing all correct partial fraction coefficients. Meanwhile, 2-3 marks in (b) were awarded for providing correct steps in solving the problem and the correct partial fraction coefficients for each fraction obtained.

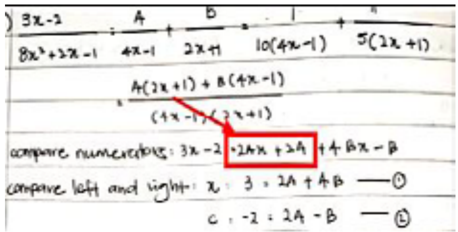
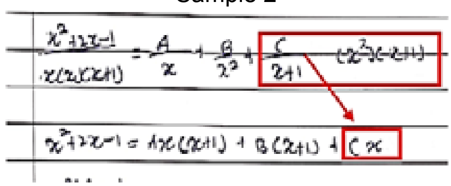
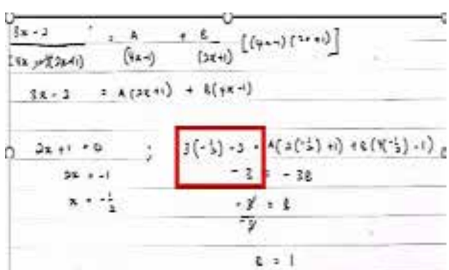
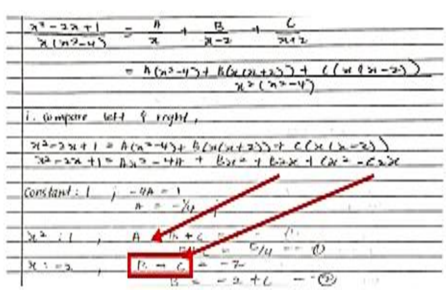
Additionally, at the end of each quiz, students were asked to fill a survey form comprising two self-evaluation questions. These questions were adopted from Man (2009). This is to measure their level of confidence in solving each problem and their perception on the degree of difficulties. Their responses were recorded on a five-point Likert scale (1 = very unconfident to 5 = very confident; 1 = very easy to 5 = very difficult) for subsequent analyses. The confidence rating above was used to filter unanswered and conjecture solutions to ensure the quality of the data collected. The rating of 1 and 5 with incomplete or blank solutions were removed. Across the filtering, from a total of 55 responses, 8 were excluded from the data set. This process of filtering was based on the guidelines provided by Krosnick (2018) as well as Ruhl, Belward and Balatti (2011).

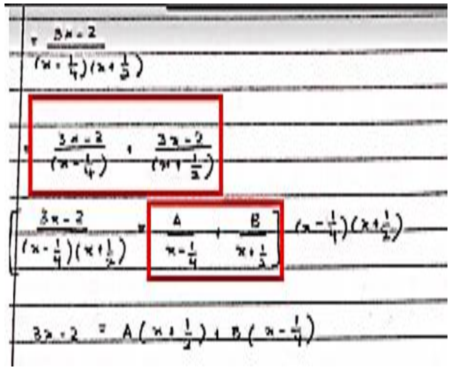
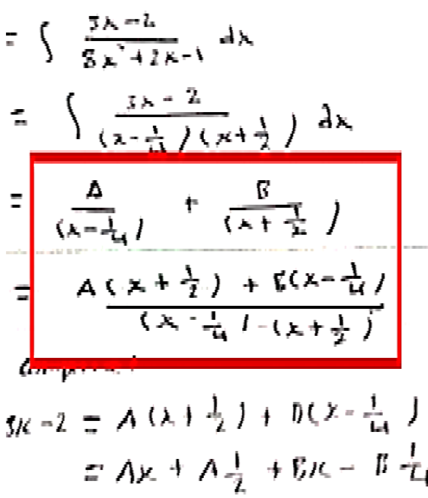
Subsequently, a questionnaire adapted from Man (2009) was used to gather students' views on the following questions: (1) Which method (I /II) is easier to understand PFD? ; (2) Which method (I /II) is easier to be used for finding PFD?; (3) Should the alternative PFD approach be introduced to students?; and (4) Which method (I /II) is more interesting for solving PFD?. Next, descriptive methods were used to analyse the common errors committed, the self-evaluation questions and the 4 items in the questionnaire. Finally, quantitative data were used to establish statistical significance of student's performance.

RESULTS AND DISCUSSIONS

The results of statistical analyses of this study are presented in the following sections. The sample data consisted of 94 responses for each problem in each quiz. Therefore, the final analysis comprised a total of 188 problems from the 94 responses. The various categories of common errors were identified from the students' responses. These errors were categorized based on the literature reviewed (e.g. Tian & Siegler, 2018; Zulfa, Suryadi, Fatimah, & Jupri, 2019; Nuri & Derya, 2017; Lynn, Amelia, Robin, Jessica, & Wang, 2017; Sheryl, 2018; Iddrisu, Abukari, & Boakye, 2017). As shown in Table 1, the percentages of these errors were tabulated. Three prominent error types of responses using Method I were discussed. The mean percentages for errors were computed from the percentages of average number of error responses on the 94 responses for each error category. For example, the mean percentage for error category No. 1 obtained was 11% as shown in Table 1 below.

Table 1: Samples of Common Errors Made when Using Method I

No.	Errors Categories	Student Samples	Mean percentage errors
1	Errors of incorrect application of the distributive law in the parenthesis	Sample 1  Sample 2 	11%
2	Errors of solving the system equation	Sample 3  Sample 4 	23%

3	Errors of factorisation	Sample 5  Sample 6 	11%

Errors of Incorrect Application of the Distributive Law in the Parenthesis

This is one of the common errors that students made when solving the expressions. As shown in Table 1, students applied 11% of inappropriate distributive law errors across the problems. They multiplied part of the brackets such as expanded A($2x+1$) to give 2Ax + 2A. Besides, they forgot to change the sign and ignored the variable after multiplying the brackets, especially for the second term in the brackets, for instance (C/(x + 1))(x + 1)(1 - x^2) as Cx. They assumed the presence of brackets as a sign to multiply and failed to concentrate on the operations adjacent to the brackets. Hence, the solution provided was incomplete. This can be attributed to deficient solving skills, lack of critical thinking, and carelessness (Ragma, 2014).

The above finding is aligned with the study of Egodawatte (2011). He advocated that students were poor in simplification, performing operations, applying exponential laws in factoring, product patterns, incorrect distribution, and invalid cancellation. He revealed that students applied transformation and processing wrongly in algebra expressions, factoring and special products of word problems. Moreover, he found that students failed to remember and correctly apply the special product and factoring patterns. He explained that students had difficulty in carrying out several steps involved in the mathematical process which led to errors. Additionally, this finding also supported Norton and Irvin's (2011) results which revealed that the main factors leading to these errors were caused by lack of or incomplete understanding of arithmetical concepts or failure to transfer arithmetic understandings to algebraic context.

Errors of Solving the System Equation

Another common error that students made was putting the wrong sign when moving terms to another position in the equation. From Table 1, it is noticed that most of the students committed 23% of this error category. They were confused when applying arithmetic operation while rearranging the equations. They also had the tendency to remove terms partially when rearranging the equation. For example, B2x – C2x become B + C or they would remove terms from inside a bracket before the bracket had been expanded when rearranging equations, for instance, 3(–1/2) –2 to –3. Sometimes, due to carelessness, they also incorrectly copied the given values and failed to complete the questions. Furthermore, students also applied wrongly and misinterpreted the two standard methods for solving linear system equations. They assumed one variable was the subject of one equation and had problems to carry out proper substitution and expansion in the other equation. Furthermore, it is observed that even when they had correctly applied substitution and expansion, they misused the "change-side, change-sign" rule. They mainly focused on the numbers and the variables without paying due attention to signs and operations. It reflects inappropriate criteria for defining an equation and incorrect description of the objects. This finding is consistent with the study done by Clement (2002). His study indicates that students failed to simplify the answers in the problems although they had correct answers. He further described that students had insufficiency of skills or knowledge pertaining to how certain variables were handled or how certain equation algorithms were processed which eventually led to transformation and processing errors on systems of equations.

Errors of Factorisation

Error of factorisation is another mistake commonly made by students. In Table 1, students committed 11% of failure in finding factors and factorising partial terms in the expressions. They were confused about the meaning of expressions and equations. They used the quadratic formula to find the roots of equations when factorising. For instance, $(4x - 1)(2x + 1)$ becomes $(x - \frac{1}{4})(x + \frac{1}{2})$ after factorising. Meanwhile, they failed to use the correct formula in expressions or equations. Also, it is observed that students sometimes failed to recognize the common factors in the numerator, which led to the cancellation of the expressions, both for the numerator and denominator. This is due to insufficient mastery in factorisation. This finding is in line with Ashlock's (2010) study. He explained that students with poor mathematical performance often ignored expressions which were lengthy and contained complex expressions and exponents. The researchers of this study also agreed that these errors are caused by students' high anxiety and poor exposure to such kind of problem.

With reference to the above discussions on the most common errors identified in this study, the findings in Table 1 indicate that students committed an average of 45% errors in solving PFD for each problem. This prevents students from obtaining better score which eventually leads to poor performance. Hence, improved cover-up method (Man, 2009) was chosen for this study which requires simple substitutions and polynomial divisions in solving PFD. It is aimed at reducing the percentage of the common errors emerged in solving PFD by the undetermined coefficient method. Subsequently, this method could help to improve students' understanding and performance in PFD. This study is being driven by Booth, Barbieri, Eyer, & Paré-Blagoev (2014). He suggested that "the misconceptions underlying specific persistent errors are not corrected through typical instruction and may require additional intervention for students to learn proper strategies" (p. 21). Thus, the focus of the next section is to study the impact of Method I and Method II on students' performance in PFD and confirm the effectiveness of Method II in assisting students to improve their performance in PFD. The discussions of the analyses are based on the results of the quizzes using both Method I and Method II performed by the students involved in this study.

Statistical Analyses and Results of the Quizzes

To examine the students' performance, the paired t-test was used. The mean scores (M) using these two methods in solving each problem by the students in both class A and B were obtained. The significant differences in the mean scores were tested to measure the 'calculation accuracy and mastery of the method used' as well as 'number of correct partial fraction coefficients' at 5% significant level. The degree of differences in mean scores obtained using both methods was examined by performing effect size. Normality was checked by using the Shapiro-Wilk test and Wilcoxon Signed Ranks Test. Lastly, the summary of students' performance using each method, which is the average scores obtained for each problem in the quizzes was presented.

Table 2: Mean and Standard Deviation to Gauge Calculation Accuracy and Mastery of the Methods Used

Class	Partial Fractions	Method	N	Mean	SD	t	Effect Size
A	Two	I	25	2.28	0.96	–2.07*	–0.41
		II	25	2.64	0.63		
	Three	I	25	2.28	0.96	–2.26*	–0.45
		II	25	1.51	0.94		
B	Two	I	22	1.91	1.48	–1.74	–0.55
		II	22	2.59	0.91		
	Three	I	22	1.14	1.36	–1.77	–0.43
		II	22	1.73	1.39		

*5% significant level

Table 3: Mean and Standard Deviation to Gauge the Number of Correct Partial Fraction Coefficients

Class	Partial Fractions	Method	N	Mean	SD	t	Effect Size
A	Two	I	25	5.32	2.73	–2.06*	–0.58
		II	25	6.52	1.08		
	Three	I	25	2.12	2.86	–2.69	–0.63
		II	25	3.68	2.01		
B	Two	I	22	5.00	2.74	–1.82	–0.56
		II	22	6.27	1.70		
	Three	I	22	3.27	2.73	–1.51	–0.36
		II	22	4.32	3.06		

*5% significant level

Table 4: Summary of Students' Performance in the Quizzes

Class	Partial Fractions	Method	Scores (%)
A	Two	I	76
		II	92
	Three	I	29
		II	49
B	Two	I	69
		II	89
	Three	I	44
		II	60

Table 5: Mean Scores of Overall Student's Performances for Methods I and II with Respect to Two and Three Partial Fractions

Partial Fractions	Method	N	Mean	SD	t	p	Effect Size
Two	I	47	7.13	3.66	–2.94*	0.005	–0.43
	II		9.02	2.29			
Three	I	47	3.62	3.85	–2.60	0.012	–0.38
	II		5.45	3.60			

*5% significant level

As shown in Table 2, there is a significant difference between the mean scores obtained by students in class A for Method I and Method II in calculation accuracy and mastery of the methods used with respect to two [$t(24) = –2.07$, $p < 0.05$; effect size = –0.41] and three partial fraction coefficients [t(24) = –2.26, $p < 0.05$; effect size = –0.45]. Likewise, there is a significant difference in the mean scores using both methods in finding number of correct two [t(24) = –2.06, $p < 0.05$; effect size = –0.58] and three [t(24) = –2.69, $p < 0.05$; effect size = –0.63] partial fraction coefficients performed by students in class A as shown in Table 3. These results suggest that students have greater understanding of the concept in applying Method II compared to Method I as well as to obtain to a larger amount of correct partial fraction coefficients when they choose to use Method II instead of Method I. Although the difference between mean scores obtained by students in class B for Method I and Method II as displayed in Table 2 and 3 is not significant at a level of significance of 0.05, there has been an increase in students' performance in applying Method II compared to Method I. The effect sizes of these analyses are found to be in a range of moderate-to-large. This indicates that students in class B would obtain mean scores for Method II at above 65^{th} percentile of the mean scores for Method I (Cohen, 1988 cited in Kraft, 2020; Matthew, 2019; Nicholas, 2017).

The overall effect sizes in Table 2 and 3 are in the range of moderate-to-large according to Cohen's conventions (Cohen, 1988 cited in Kraft, 2020; Matthew, 2019; Nicholas, 2017). As stated by these authors, the negative magnitude of these effect sizes indicate that students who apply Method II have higher mean scores in regard to the 'calculation accuracy and mastery of the method used' as well as 'number of correct partial fraction coefficients' compared to students who apply Method I. This finding suggests that there are differences between the students who apply Method I and Method II in solving partial fraction. Besides, the results indicate that students who used Method II have better scores performance than those students who used Method I.

Generally, the results in Table 2 and 3 show that the mean scores for Method II are larger whereas the standard deviations are smaller compared to that for Method I. This implies that Method II could significantly affect students' performance in solving PFD. These results are consistent with the average performance in each class as depicted in Table 4. Moreover, it is interesting to note that the difference between the means scores obtained for Method I and Method II in solving three partial fraction coefficients are higher in class B as compared to class A, implying that even though there is no significant difference in the means scores of each category in Table 2 [t(21) = –1.77, $p > 0.05$; effect size = –0.43] and Table 3 [t(21) = –1.51, $p >$

0.05; effect size = –0.36], students' performances are significantly increased when Method II is used to decompose partial fraction. This finding is coherent with the results of the self-evaluations which indicate that students find the use of Method II in solving problem to be beneficial. This result is also supported by 89% of the students who have stated no difficulty in solving the problem and 91% of the students indicate the confidence of getting correct partial fraction using Method II compared to Method I.

As a supplement to the results in Table 4, Table 5 shows that, with respect to students' performance in two partial fractions, it is statistically significant [t(46) = –2.94, $p < 0.05$; Wilcoxon Signed Ranks Test < 0.05; effect size = 0.43; test power = 0.92] between Method I (M = 7.12, SD = 3.66) and Method II (M = 9.02, SD = 2.89). Similarly, the results with regard to students' performance in three partial fractions show a significant difference in the mean scores for Method I (M = 3.62, SD = 3.85) and Method II (M = 5.45, SD = 3.60); t(46) = –2.60, $p < 0.05$; Shapiro-Wilk test > 0.05; effect size = 0.38; test power = 0.92). The effect sizes for both analyses indicate a range of medium to large (Cohen, 1988 cited in Kraft, 2020; Matthew, 2019; Nicholas, 2017). Therefore, it can be concluded that the use of Method II in solving PFD has positive impact on students' performance compared to that of Method I.

Students' Opinions on Learning and Applying Method I and Method II

Table 5: Summary of the Students' Responses to Using Method I and Method II

Class	Classification	Method I	Method II
A	understanding	36% (9)	64% (16)
	ease of use	36% (9)	64% (16)
	suitability	28% (7)	72% (18)
	interesting	52% (13)	48% (12)
B	understanding	18% (4)	82% (18)
	ease of use	14% (3)	86% (19)
	suitability	14% (3)	86% (19)
	interesting	18% (4)	82% (18)

As presented in Table 5, generally, the percentage of favouring Method II is higher than Method I in both classes A and B on four aspects, namely understand the concept of the methods, ease of using the methods in calculations, suitability of the methods to be introduced, and the interest in using the methods concerned. Hence, it can be said that in terms of understanding the concepts for both methods, students perceived Method II as easier to be understood than that of Method I. This could be due to the reason that Method II was taught with more demonstration of examples by the researcher of this study since it is new to them. It is believed that more discussions on the practices given by the researcher had motivated students to use Method II in the assessments.

Besides, 64% of the students in class A and 86% of the students in class B agree that Method II is easier to be used due to the fewer computation steps involved when finding three partial fraction coefficients as compared to using Method I. Students who are not proficient with systems of equations report that they have difficulties in solving three unknown variables in the systems of equations. This finding is reflected in the scores presented in Table 4.

The findings of this study also show that more than 70% of the students in both classes agree that Method II should be taught. As presented in Table 4, it is evident that their performance of PFD is better when they apply Method II. 82% of the class B students find that Method II is more interesting than Method I. However, it is noticed that 52% of the students in class A prefer Method I to Method II. But interestingly, students in class A have achieved better performance when using Method II as shown in Table 2, 3 and 4 although they indicate their preference for Method I. This proves the effectiveness of using Method II in solving PFD coefficients.

CONCLUSION AND RECOMMENDATIONS

To meet the objectives of this study, specifically to explore and determine the potential application of improved cover-up method on teaching undergraduate students as an effective alternative method to solve PFD coefficients problems, three prominent errors committed by students when performing the computations in finding PFD coefficients using the undetermined coefficient method (Method I) were identified. An alternative partial decomposition method, namely, the improved cover-up method (Method II) was then introduced to explore its potential in helping students perform better when solving PFD problems. The analyses of the findings obtained in this study have shown the promising potential of Method II in view of the progress made in the students' performance after having learnt this method, particularly in terms of calculation accuracy and mastery of the method, and obtaining more correct answers in solving problems related to PFD coefficients. The findings also indicate that students who attained grades C to B+ in Calculus I were benefited from the use of Method II in solving PFD. Thus, it is believed that Method II can serve as a more sustainable alternative in teaching undergraduate students the various elementary integral calculus courses. Moreover, 70% of the respondents in this study have agreed that Method II should be taught to the class. This agreement has been quantified and is reflected in students' improved performance when they apply Method II compared to Method I in solving PFD problems.

Finally, this study provides preliminary evidence that Method II could address the identified common errors made when applying standard method (Method I) and hence, Method II is recommended to teach undergraduate's elementary integral calculus courses. However, there are certain pedagogical perspectives need to be addressed particularly, the lengthy effort spent on teaching and learning the concepts behind Method II. Therefore, future research which focuses on how to teach or learn this new approach to solving other forms of PFD related problems is needed. Furthermore, future research could also consider larger sample size involving participants from different universities as well as complemented by formal cognitive interview report.

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